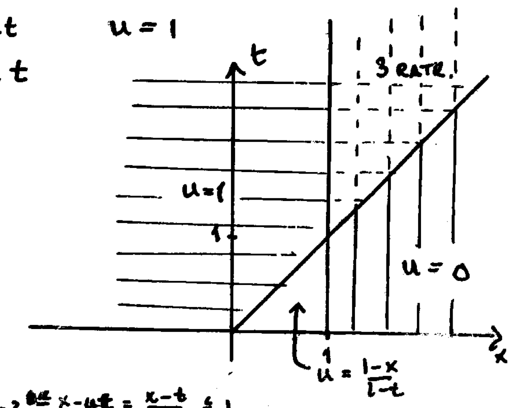
…AALTOYHTÄLÖ (JATK.)

$$u_t + u u_x = 0$$

$u(x, t=0) = f(x)$ $\qquad u(x,t) = f\left(x - u(x,t)\cdot t\right)$

ESIM. $f(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$ $\qquad$ MIKÄ ON $u(x,t)$ KUN $t > 0$?

1° $x - ut \le 0 \Leftrightarrow x \le ut$ $\qquad u = 1$
ELI $x \le t$



2° $x - ut \ge 1$ $\qquad u = 0$
ELI $x \ge 1$

3° $0 \le x - ut \le 1$

$u = 1 - x + ut$

$\Leftrightarrow \; u = \dfrac{1-x}{1-t}$

ALUE? $0 \le x - ut = \frac{x-t}{1-t} \le 1$

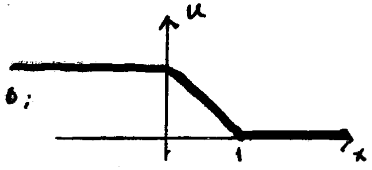
$0 < t < 1 \Rightarrow 0 \le x - t \le 1 - t$

ELI $x \ge t$ & $x \le 1$

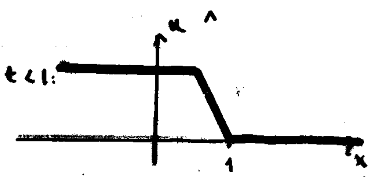
$t > 1$

$0 \ge x - t \ge 1 - t$ ELI $x \le t$ & $x \ge 1$

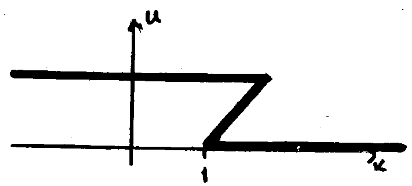
$t = 0$;



$0 < t < 1$:



$t > 1$:



2. TOISEN KERTALUVUN ODY:t

ESIMERKKI: YKSIULOTTEINEN AALTOYHTÄLÖ $\quad u = u(x,t)$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

OTAMME KÄYTTÖÖN UUDET MUUTTUJAT

$$\xi = x - ct \quad , \quad \eta = x + ct$$

S.O. $\quad x = \frac{1}{2}(\xi + \eta) \quad , \quad t = \frac{1}{2c}(\eta - \xi)$

MINKÄNÄKÖINEN ON (1) NÄISSÄ MUUTTUJISSA?

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = c\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}\right) - c^2 \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}\right)$$

$$= c^2\left[\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \xi^2}\right]$$

$-\infty < x < \infty$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$(1) \Leftrightarrow \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

YLEINEN RATKAISU ON HELPOSTI ARVATTAVISSA:

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

f, g MIELIVALTAISIA KAHDESTI DIFFERENTIOITUVIA

$$u(x,t) = f(x-ct) + g(x+ct) \qquad (2)$$

RATKAISU SAADAAN YKSIKÄSITTEISEKSI ASETTAMALLA ALKUEHTOJA

CAUCHYN ALKUEHDOT:

$$u(x,0) = \varphi(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv u_t(x,0) = \psi(x)$$

$\left(u_{xx} = \frac{\partial^2 u}{\partial x^2},\ u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \text{ JNE.}\right)$

SIJOITETAAN (2):EEN:

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$u(x,0) = \varphi(x) \qquad u_t(x,0) = \psi(x)$$

$$\Rightarrow \quad f(x) + g(x) = \varphi(x) \qquad (3)$$

$$-cf'(x) + cg'(x) = \psi(x) \qquad (4)$$

(4) VOIDAAN INTEGROIDA:

$$f(x) - g(x) = -\frac{1}{c}\int_0^x dx'\,\psi(x') + \underbrace{f(0) - g(0)}_{K}$$

SIIS:

$$f(x) + g(x) = \varphi(x)$$

$$f(x) - g(x) = -\frac{1}{c}\int_0^x dx'\,\psi(x') + K$$

$$\Rightarrow f(x) = \frac{1}{2}\left[\varphi(x) - \frac{1}{c}\int_0^x dx'\,\psi(x') + K\right]$$

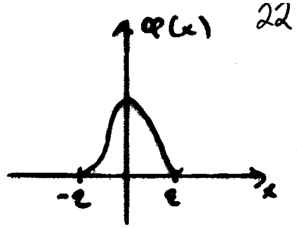
$$g(x) = \frac{1}{2}\left[\varphi(x) + \frac{1}{c}\int_0^x dx'\,\psi(x') - K\right]$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} dx'\,\psi(x')\right]$$

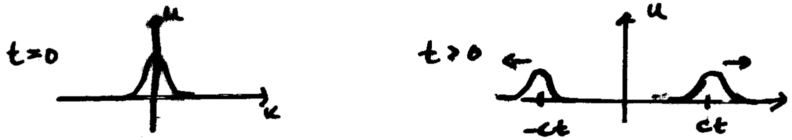
1-KÄS. RATKAISU (D'ALEMBERT)

TAPAUS $\psi(x) = 0$

OL. $\varphi(x) \neq 0 \quad x \in [-\varepsilon, \varepsilon]$
("häiriö")

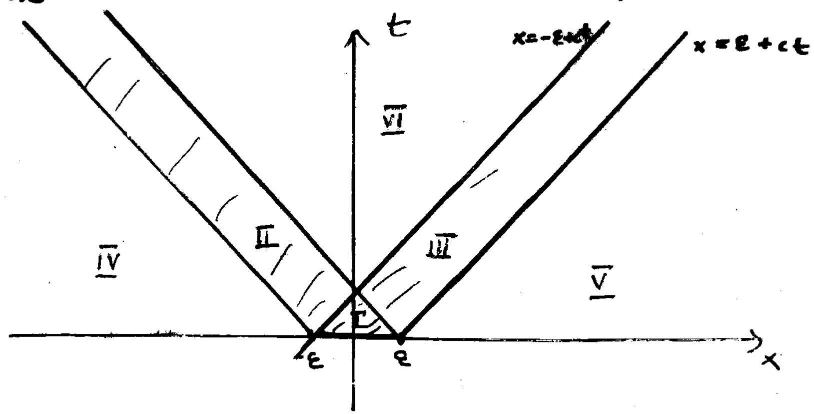


$\Rightarrow \quad u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right)$



HÄIRIÖ LIIKKUU KARAKTERISTIKOITA PITKIN

$\xi = x - ct = \text{vakio}$
$\eta = x + ct = \text{vakio}'$



ALUEESSA I SEKÄ ETEEN- ETTÄ TAAKSEPÄIN ETENEVÄT AALLOT VAIKUTTAVAT

II AINOASTAAN TAAKSEPÄIN ETENEVÄ AALTO

III — " — ETEENPÄIN — " —

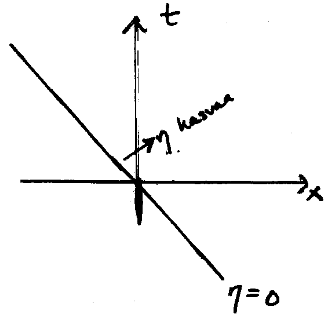
IV, V $u=0$, AALTO EI VIELÄ EHTINYT VAIKUTTAA

VI AALLOT OHITTANEET, $u=0$ TAAS

(TAPAUS $\varphi=0$, $\psi \neq 0$ HARJ. TEHT.)

HUONOT ALKUEHDOT:

$$u(\xi, \eta=0) = \varphi(\xi) \qquad (A)$$
$$\frac{\partial u}{\partial \eta}(\xi, \eta=0) = \psi(\xi) \qquad (B)$$



$u(\xi,\eta) = f(\xi) + g(\eta)$

(A) $\Rightarrow f(\xi) + g(0) = \varphi(\xi)$

(B) $\Rightarrow g'(0) = \psi(\xi) \qquad (B')$

(B') OK VAIN JOS $\psi = \text{VAKIO} \equiv \psi_0$

EI 1-KÄS RATKAISUA:

$$u(x,t) = \varphi(x-ct) + g(x+ct) - g(0)$$

MISSÄ $g(x)$ MUUTEN MIELIVALTAINEN PAITSI $g'(0) = \psi_0$

JOS $\psi(x) \neq$ vakio RATKAISUA EI OLE

ALKUEHTOJA EI SAA ANTAA KARAKTERISTIKALLA!

AALTOYHTÄLÖN RATKAISU FOURIERIN MUUNNOKSELLA

$-\infty < x < \infty$; $u(x,t) \to 0$ $|x| \to \infty$ 'REUNAEHTO'

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, e^{i(kx - \omega t)} \, a(k,\omega)$$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \, e^{i(kx-\omega t)} \left(-\frac{\omega^2}{c^2} + k^2\right) a(k,\omega)$$

$$= 0$$

$$\left(k^2 - \frac{\omega^2}{c^2}\right) a(k,\omega) = 0 \quad \text{ELI} \quad \left(k + \frac{\omega}{c}\right)\left(k - \frac{\omega}{c}\right) a(k,\omega) = 0$$

MUISTA : $x\,\delta(x) = 0$

$$\Rightarrow \frac{a(k,\omega)}{\sqrt{2\pi}} = \delta(\omega - ck)\,A(k) + \delta(\omega + ck)\,B(k)$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[A(k)\, e^{i(kx - ckt)} + B(k)\, e^{i(kx+ckt)}\right]$$

$$= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left[A(k)\, e^{ik(x-ct)} + B(k)\, e^{ik(x+ct)}\right]$$

YLEINEN RATKAISU

CAUCHY: $u(x,0) = \int \frac{dk}{\sqrt{2\pi}} \left(A(k) + B(k)\right) e^{ikx} = \varphi(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{ikx}\, \tilde{\varphi}(k)$$

$$\Rightarrow \quad A(k) + B(k) = \tilde{\varphi}(k)$$

$$u_t(x,0) = ic \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \, k\left(-A(k) + B(k)\right) e^{ikx}$$

$$= \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \tilde{\psi}(k)\, e^{ikx}$$

$$\Rightarrow \quad ick\left(-A(k) + B(k)\right) = \tilde{\psi}(k)$$

SIIS
$$A(k) + B(k) = \tilde{\varphi}(k)$$
$$A(k) - B(k) = \frac{i}{c} \frac{\tilde{\psi}(k)}{k}$$

$$\Rightarrow A(k) = \frac{1}{2}\left(\tilde{\varphi}(k) + \frac{i}{c}\frac{\tilde{\psi}(k)}{k}\right)$$

$$B(k) = \frac{1}{2}\left(\tilde{\varphi}(k) - \frac{i}{c}\frac{\tilde{\psi}(k)}{k}\right)$$

MÄÄRÄÄ $u(x,t)$:n 1-KÄS.

FOURIERIN MUUNNOS MUUTTAA LINEAARISEN ODY$_n$ (TAI DY$_n$) ALGEBRALLISEKSI YHTÄLÖKSI

AALTOYHTÄLÖ ÄÄRELLISELLÄ VÄLILLÄ

RATKAISTAVA $u_{tt} - c^2 u_{xx} = 0 \quad 0<x<L,\ 0<t<\infty$

ALKUEHDOT (CAUCHY) $u(x,t=0) = \varphi(x) \quad 0<x<L \quad (A1)$

$u_t(x,t=0) = \psi(x) \quad (A2)$

NYT ON LISÄKSI ASETETTAVA REUNAEHTOJA $x=0,\ x=L$:SSÄ, $0<t<\infty$

ESIM. $u(0,t) = u(L,t) = 0 \quad 0<t<\infty$

(VÄRÄHTELEVÄ KIELI, PÄÄT KIINNITETTY)

KUN REUNAEHDOT JAKSOLLISIA, VOIDAAN KEHITTÄÄ $u(x,t)$ FOURIER'N SARJAKSI x:SSÄ; ODY $\rightarrow$ TAVALLISET DY:T KERTOIMILLE (YO. ESIMERKKITAPAUKSESSA SINISARJA $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$ REUNAEHTOJEN KANSSA SOPIVA; HARJ.)

OTETAAN TÄSSÄ MONIMUTKAISEMPI ESIMERKKI

REUNAEHDOT: $u(0,t) = 0 \quad 0<t<\infty \quad (R1)$

$u_x(L,t) = 0 \quad (R2)$

(VÄRÄHTELEVÄ KIELI; PÄÄ $x=0$ KIINNITETTY, PÄÄ $x=L$ VAPAA)

$u_{tt} - c^2 u_{xx} = 0 \quad 0<x<L \quad 0<t<\infty \quad (Y)$

$u(x,0) = \varphi(x) \quad (A1)$

$u_t(x,0) = \psi(x) \quad (A2)$

$u(0,t) = 0 \quad (R1)$

$u_x(L,t) = 0 \quad (R2)$

RATKAISTAAN MUUTTUJIEN EROTTELULLA:

ETSITÄÄN RATKAISU (Y):LLE, JOKA TOTEUTTAA (R1) & (R2), MUOTOA $u(x,t) = X(x)T(t)$

$(Y) \Rightarrow \ddot{T}(t)X(x) = c^2 T(t) X''(x)$

$\left(\ddot{T} = \frac{d^2T}{dt^2}\right) \quad \left(X'' = \frac{d^2X}{dx^2}\right)$

JAETAAN $u = XT$:LLÄ $\Rightarrow$

$$\frac{\ddot{T}}{T} = c^2 \frac{X''}{X} = -\lambda^2$$

RIIPPUU VAIN t:STÄ — RIIPPUU VAIN x:STÄ $\Rightarrow$ VAKIO (VAL. <0; HALUTAAN VÄRÄHTELYÄ!)

$\Rightarrow X'' = -\left(\frac{\lambda}{c}\right)^2 X \quad X(0)=0 \quad X'(L)=0$

$X(x) = A\sin\left(\frac{\lambda x}{c}\right) + B\cos\left(\frac{\lambda x}{c}\right)$

$X(0)=0 \Rightarrow B=0$

$X'(L)=0 = \frac{\lambda A}{c}\cos\left(\frac{\lambda L}{c}\right) = 0$

$\Rightarrow \lambda = \lambda_n = \frac{c}{L}(2n+1)\frac{\pi}{2} = \frac{c}{L}\left(n+\frac{1}{2}\right)\pi \quad n=0,1,2,\ldots$

VAIN TIETYT ARVOT ("OMINAISARVOT") KELPAAVAT!

3-UL. AALTOYHTÄLÖ R^3:SSA

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\vec{x},t) = 0 \qquad u(\vec{x},t) \underset{|\vec{x}|\to\infty}{\to} 0$$

$$u(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, a(\vec{k},\omega)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d\omega}{\sqrt{2\pi}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \left(-\frac{\omega^2}{c^2} + \vec{k}^2\right) a(\vec{k},\omega)$$

$$\Rightarrow (\omega^2 - c^2\vec{k}^2)\, a(\vec{k},\omega) = (\omega - c|\vec{k}|)(\omega + c|\vec{k}|)\, a(\vec{k},\omega) = 0$$

$$\Rightarrow a(\vec{k},\omega) = \sqrt{2\pi}\left[\delta(\omega - c|\vec{k}|)A(\vec{k}) + \delta(\omega + c|\vec{k}|)B(\vec{k})\right]$$

- YLEINEN RATKAISU:

$$u(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\left[e^{-ic|\vec{k}|t} A(\vec{k}) + e^{ic|\vec{k}|t} B(\vec{k})\right]$$

$A(\vec{k})$, $B(\vec{k})$ MIELIVALTAISIA

ALKUEHDOT (CAUCHY)

$$u(\vec{x},0) = \varphi(\vec{x})$$

$$\frac{\partial u}{\partial t}(\vec{x},0) \equiv \partial_t u(\vec{x},0) = \psi(\vec{x})$$

MÄÄRÄÄVÄT $A(\vec{k})$, $B(\vec{k})$:N $\Rightarrow$ YKSIKÄSITTEINEN RATKAISU

"OMINAISFUNKTIOT"

$$\Rightarrow X_n(x) = A_n \sin\left((n+\tfrac{1}{2})\pi \frac{x}{L}\right) \qquad n = 0, 1, 2, \ldots$$

($n<0$ EI ANNA UUSIA RATK.)

VASTAAVA $T_n(t) = C_n \sin(\lambda_n t) + D_n \cos(\lambda_n t)$

(Y):N JA (R1) & (R2):N YLEINEN RATKAISU ON

$$u(x,t) = \sum_{n=0}^{\infty}\left(a_n \sin\left[(n+\tfrac{1}{2})\frac{\pi c t}{L}\right] + b_n \cos\left[(n+\tfrac{1}{2})\frac{\pi c t}{L}\right]\right) \times \sin\left[(n+\tfrac{1}{2})\frac{\pi x}{L}\right]$$

($a_n = A_n C_n$, $b_n = A_n D_n$; VAIN KAKSI RIIPPUMATONTA KERROINTA)

KERTOIMET a_n, b_n MÄÄRÄYTYVÄT ALKUEHDOISTA

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\left[(n+\tfrac{1}{2})\frac{\pi x}{L}\right] = \varphi(x) \qquad (A1)$$

$$u_t(x,0) = \sum_{n=0}^{\infty}\left(+c(n+\tfrac{1}{2})\frac{\pi}{L}\right) a_n \sin\left[(n+\tfrac{1}{2})\frac{\pi x}{L}\right] = \psi(x) \qquad (A2)$$

KÄYTTÄEN ($\xi = \frac{x}{L}$, DIMENSIOTON MUUTTUJA)

$$\int_0^1 d\xi \,\sin\left[(n+\tfrac{1}{2})\pi\xi\right] \sin\left[(m+\tfrac{1}{2})\pi\xi\right] = \frac{1}{2}\delta_{mn} \qquad \text{JOHDA!}$$

"OMINAISFUNKTIOIDEN ORTOGONAALISUUS"

SAADAAN

$$b_n = \frac{2}{L}\int_0^L dx\, \varphi(x) \sin\left[(n+\tfrac{1}{2})\frac{\pi x}{L}\right]$$

$$a_n = +\frac{2}{c(n+\frac{1}{2})\pi}\int_0^L dx\, \psi(x) \sin\left[(n+\tfrac{1}{2})\frac{\pi x}{L}\right]$$

(Arfken-Weber 15.4) 30

TOINEN ESIMERKKI:

DIFFUUSIOYHTÄLÖ

$$-\infty < x < \infty, \; 0<t<\infty$$
$$u(x,t) \to 0 \quad |x|\to\infty$$

$$\boxed{\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{ikx}\, a(k,t)$$

$$u_t - D u_{xx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{ikx}\left(\frac{\partial a}{\partial t} + Dk^2 a\right) = 0$$

$$\Rightarrow \frac{\partial a}{\partial t} = -Dk^2 a \quad \Rightarrow \boxed{a(k,t) = e^{-Dk^2 t}\, a(k,0)}$$

ALKUEHTO: $u(x,0) = \varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \tilde{\varphi}(k)$

$$\Rightarrow a(k,0) = \tilde{\varphi}(k)$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{ikx - Dk^2 t}\,\tilde{\varphi}(k)}$$

HUOM! RATKAISU PYSYY RAJOITETTUNA ("ON STABIILI")
VAIN JOS $Dt > 0$ $|t|\to\infty$ ELI

$D>0 \quad t>0 \quad t\to\infty$ (FYSIKAALINEN TAPAUS)

$D<0 \quad t<0 \quad t\to-\infty$

DIFFUUSIOYHTÄLÖ KUVAA IRREVERSIIBELIÄ PROSESSIA

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$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{ikx - Dk^2 t}\,\tilde{\varphi}(k)$$

OTETAAN $u(x,0) = S\,\delta(x) = \varphi(x)$
("PISTEMÄINEN LÄHDE")

MUISTA: $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ikx}$

$$\Rightarrow \tilde{\varphi}(k) = \frac{S}{\sqrt{2\pi}}$$

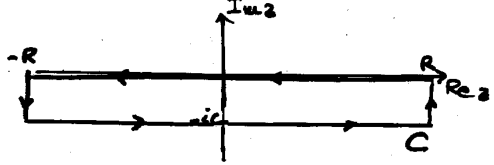
$$u(x,t) = \frac{S}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ikx - Dtk^2} = \frac{S}{2\pi}\int_{-\infty}^{\infty} dk\; e^{-\left(\sqrt{Dt}\,k - \frac{ix}{2\sqrt{Dt}}\right)^2} \times e^{-\frac{x^2}{4Dt}}$$

UUSI MUUTTUJA $z = \sqrt{Dt}\,k - \frac{ix}{2\sqrt{Dt}} = \sqrt{Dt}\,k - ic$

$$dk = \frac{1}{\sqrt{Dt}}\,dz$$

$$u(x,t) = \frac{S\,e^{-x^2/4Dt}}{2\pi\sqrt{Dt}}\int_{-\infty-ic}^{\infty-ic} dz\; e^{-z^2} = \frac{S}{\sqrt{4\pi Dt}}\,e^{-\frac{x^2}{4Dt}}$$

KS. ALLA



$$\oint_C dz\, e^{-z^2} = 0 = \int_{-R-ic}^{R-ic} dz\, e^{-z^2} + \int_{R}^{-R} dz\, e^{-z^2} + i\int_{-c}^{0} dy\, e^{-(R+iy)^2}$$

$R\to\infty$: $\to 0$

$$+\, i\int_{0}^{-c} dy\, e^{-(-R+iy)^2} \Rightarrow \int_{-\infty-ic}^{\infty-ic} dz\, e^{-z^2} = \int_{-\infty}^{\infty} dz\, e^{-z^2} = \sqrt{\pi}$$

$0 \quad R\to\infty$

$u(x,0) = S\,\delta(x-x_0)$:TAA VASTAAVASTI

$$u(x,t) = \frac{S}{\sqrt{4\pi D t}}\, e^{-\frac{(x-x_0)^2}{4Dt}}$$

ALKUARVO-ONGELMAN RAISU SADAAN

$$u(x,0) = f(x) = {}_0 f(x_0)\,\delta(x-x_0)$$

SUPERPONOIMALLA PLÄISEN LÄHTEEN RATKAISUJA ($S \to x_0$; INTEGROIDAAN ELI SUMMATAAN LÄHDETEN YLI ; LINEAARISUUS ⇒ RATKAISU = VASTAAVALLA)

$$u(x,t) = \int_{-\infty}^{\infty} dx_0\, f(x_0) \underbrace{\frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x-x_0)^2}{4Dt}}}_{\text{DIFFUUSIOYHTÄLÖN "GREENIN FUNKTIO"}}$$

(OPIMME LISÄKSI:

$$\lim_{t\to 0} \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x-x_0)^2}{4Dt}} = \delta(x-x_0) \;)$$

TOISEN KERTALUVUN ODY:DEN LUOKITTELU

$u = u(x_1, x_2, \dots, x_n) = u(x)$

$$\sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + F\left(x, u, \frac{\partial u}{\partial x_i}\right) = 0$$

KOSKA $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$ VOIMME OLETTAA $A_{ij} = A_{ji}$

(MIKSI? TARKASTELLAAN $\sum_{i,j} A_{ij} M_{ij}$, MISSÄ $M_{ij} = M_{ji}$.

KIRJOITETAAN $A_{ij} = \underbrace{\frac{1}{2}(A_{ij}+A_{ji})}_{\equiv A_{ij}^{(S)} = A_{ji}^{(S)}} + \underbrace{\frac{1}{2}(A_{ij}-A_{ji})}_{\equiv A_{ij}^{(A)} = -A_{ji}^{(A)}}$

SYMMETRINEN OSA ANTISYMM. OSA

TÄLLÖIN $\sum_{i,j} A_{ij} M_{ij} = \sum_{i,j} A_{ij}^{(S)} M_{ij} + \sum_{i,j} A_{ij}^{(A)} M_{ij}$

JÄLKIMMÄINEN TERMI

$$\sum_{i,j} A_{ij}^{(A)} M_{ij} \underset{M_{ij}=M_{ji}}{=} \sum_{i,j} A_{ij}^{(A)} M_{ji} \overset{A_{ij}^{(A)}=-A_{ji}^{(A)}}{=} -\sum_{i,j} A_{ji}^{(A)} M_{ji} =$$

$$\underset{\text{NIMETÄÄN SUMMAUSINDEKSIT UUDELLEEN } i\to j,\ j\to i}{=} -\sum_{i,j} A_{ij}^{(A)} M_{ij} = 0 \;)$$