

YHTYYS KVAANTIMEKANIIKKAAN

1. FYSIIKKA OPERAATTORI $(\hat{X}, D_{\hat{X}})$

$$(\hat{X}\psi)(\omega) = x\psi(\omega) \quad \psi \in L^2(\mathbb{R})$$

$$D_{\hat{X}} = \{ \psi \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} dx |\omega\psi(\omega)|^2 < \infty \}$$

$(\hat{X}, D_{\hat{X}})$ ITSEADJUNGOITU

SPEKTRALIISITYYS:

MÄÄR. OPERAATTORIT \hat{E}_x $-\infty < x < \infty$

$$(\hat{E}_x\psi)(y) = \begin{cases} \psi(y) & y \leq x \\ 0 & y > x \end{cases} \quad -\infty < x < \infty$$

$$E_{11}(\hat{E}_x\psi)(y) = \int_{-\infty}^{\infty} dx \delta(y-x)\psi(x) \quad (1)$$

HUOM. \hat{E}_x ON PROJEKTIO-OPERAATTORI
ALIVARUUDELLE

$$\forall x = \{ \psi \in L^2 \mid \psi(y) = 0 \quad x < y \}$$

LIISÄKSI $\hat{E}_{-\infty} = \hat{0}$ NOLEAOP. $\hat{E}_{+\infty} = id$

$\{\hat{E}_x\}$ TOTUTTAVAT SPEKTRALIISITYYS-

LAUSESSA VÄÄDITUT EHDOT:

OLEKON $x' \leq x$

$$(\hat{E}_x \hat{E}_{x'}\psi)(y) = \hat{E}_x \begin{cases} \psi(y) & y \leq x' \\ 0 & y > x' \end{cases} =$$

$$= \begin{cases} \psi(y) & y \leq x' \\ 0 & y > x' \end{cases}$$

$$(\hat{E}_{x'} \hat{E}_x\psi)(y) = \hat{E}_{x'} \begin{cases} \psi(y) & y \leq x \\ 0 & y > x \end{cases}$$

$$= \begin{cases} \psi(y) & y \leq x' \\ 0 & y > x' \end{cases}$$

$$\text{SIIS } \hat{E}_x \hat{E}_{x'} = \hat{E}_{x'} \hat{E}_x = \hat{E}_{x''}$$

$$\text{MISSÄ } x'' = \min(x, x')$$

$$\text{VÄITÄNNE NYT: } \hat{X} = \int_{-\infty}^{\infty} x d\hat{E}_x$$

TÄRKOITTAEN

$$\langle \varphi | \hat{X} \psi \rangle = \int_{-\infty}^{\infty} x dx \langle \varphi | \hat{E}_x \psi \rangle$$

KAIKILLE $\varphi, \psi \in L^2(\mathbb{R})$ STABIILIN INT.

TOIANTA:

$$\langle \varphi | \hat{E}_x \psi \rangle = \int_{-\infty}^{\infty} dx \varphi^*(y) (\hat{E}_x \psi)(y)$$

$$= \int_{-\infty}^{\infty} dy \varphi^*(y) \psi(y) \equiv f(x)$$

f DIFFERENTIOITUNNA

$$f'(x) = \frac{df}{dx} = \varphi^*(x) \psi(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} x \Delta \langle \varphi | \hat{E}_x \psi \rangle = \int_{-\infty}^{\infty} x df(x)$$

$$= \int_{-\infty}^{\infty} x f'(x) dx = \int_{-\infty}^{\infty} \varphi^*(x) x \psi(x) dx$$

$$= \int_{-\infty}^{\infty} dx \varphi^*(x) (\hat{X} \psi)(x) = \langle \varphi | \hat{X} \psi \rangle$$

OK!

SPECTRAALIESITYKESON AVULLA VOIDAN MÄÄN. OPER. $(F(\hat{X}), D(F\hat{X}))$

$$F(\hat{X}) = \int_{-\infty}^{\infty} F(x) d\hat{E}_x \quad \text{FUNCTIO}$$

$$D(F\psi) = \{ \psi \in L^2 \mid \int_{-\infty}^{\infty} |F(x)\psi(x)|^2 dx < \infty \}$$

MAJIMIILINEN

1. DIRACIN MERKINTÄTAPA

VEKTORI $|u\rangle$ "KET" \mathcal{H}^A :SSA $\langle v|$ "BRA"

\mathcal{H}^B IN DUALIIVARUUS

$$v(u) = \langle v | u \rangle \quad \left(\begin{array}{l} \text{RIESEN} \\ \text{ESITYSLAUS} \end{array} \right)$$

SKALAARITULO

OLKON \mathcal{H} SEPAROITUNNA JA

$\{|e_i\rangle\}$ ON. KANTTA: $\langle e_i | e_j \rangle = \delta_{ij}$

$|u\rangle$:N ESITYS KANUUNSSA $\{|e_i\rangle\}$:

$$|u\rangle = \sum_{i=1}^{\infty} |e_i\rangle \langle e_i | u \rangle = id_x |u\rangle$$

$$\Rightarrow id_x = \sum_{i=1}^{\infty} |e_i\rangle \langle e_i|$$

"TRIVELISYYSRELAATIO"

\hat{A} OPERAATTORI

$$\hat{A} |e_i\rangle = \sum_{j=1}^{\infty} |e_j\rangle a_{ji}$$

$$a_{ji} = \langle e_j | \hat{A} |e_i\rangle = \langle e_j | \hat{A} |e_i\rangle$$

$$\Rightarrow \hat{A} = \sum_{j,i} |e_j\rangle a_{ji} \langle e_i|$$

OPERAATTORIN MATEMATIIKESITYS

LAUSENUSTANIN JE SISÄLTÄMÄÄN

"VEKTORITTA" $|x\rangle$, $-\infty < x < \infty$

JOTKA TOTEUTTAVAT

$$\langle x|y\rangle = \delta(x-y)$$

$\psi \in L^2(\mathbb{R})$, MERKITÄÄN

$$\psi(x) = \langle x|\psi\rangle$$

TÄLLÖIN VOIMME KIRJOITTAA

$$\hat{E}_x = \int_{-\infty}^{\infty} dx |x\rangle \langle x|$$

KOSKA

$$(\hat{E}_x \psi)(y) = \langle y|\hat{E}_x \psi\rangle =$$

$$= \int_{-\infty}^{\infty} dx \underbrace{\langle y|x\rangle}_{\delta(y-x)} \underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int_{-\infty}^{\infty} dx \delta(y-x) \psi(x)$$

(VART. (1))

MON. $\hat{E}_p = id = \int_{-\infty}^{\infty} dx |x\rangle \langle x|$

"TÄYDELLISYYSRELAATIO"

NYT $d\hat{E}_x = |x\rangle \langle x|$

JA SPEKTRAALIESITYS

$$\hat{X} = \int_{-\infty}^{\infty} dx x |x\rangle \langle x| = \int_{-\infty}^{\infty} dx |x\rangle x \langle x|$$

TARKISTUS

$$(\hat{X} \psi)(y) = \langle y|\hat{X}|\psi\rangle =$$

$$= \int_{-\infty}^{\infty} dx \underbrace{\langle y|x\rangle}_{\delta(y-x)} x \underbrace{\langle x|\psi\rangle}_{\psi(x)} = y \psi(y)$$

OK!

3. IMPULSSIOPERAATTORI (\hat{P}, D_P)

$$(\hat{P} \psi)(x) = -i \frac{d\psi}{dx} \quad (k=1)$$

$$D_P = \{ \psi \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} dx |\psi'(x)|^2 < \infty \}$$

(\hat{P}, D_P) ITSEADJUNGOITU

OPERAALIESITYS RAKENUSTAN
YKSINKERTAISIMIN FOURIER'IN
MUUNOKSEN AVULLA:

MÄÄR. OPERAATTORI $\hat{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$

$$(\hat{F}\psi)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \psi(x) \equiv \tilde{\psi}(k)$$

($\hat{F}\psi$ ON TODEN $L^2(\mathbb{R})$:IN ALKIO, JOS ψ
ON. TODISTUS LÖYTYY MATEMATIIKAN
KIRJOISTA)

PLANCHERELIN KAAVA (HONKONEN):
FYHM I, KAAVA (6.16)) SAADO

$$\langle \varphi | \psi \rangle = \int_{-\infty}^{\infty} dx \varphi^*(x) \psi(x) = \\ = \int_{-\infty}^{\infty} dk \tilde{\varphi}^*(k) \tilde{\psi}(k) = \langle \hat{F}\varphi | \hat{F}\psi \rangle$$

\hat{F} ON SIS ISOMETRINEN

USÄKSI \hat{F}^{-1} ON OLEMASSA:

$$(\hat{F}^{-1}\tilde{\psi})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\psi}(k) = \psi(x)$$

(HONKONEN)

\hat{F} ON SIS UNITÄÄRIINEN :
 $\hat{F} \hat{F}^\dagger = \hat{F}^\dagger \hat{F} = id$, $\hat{F}^{-1} = \hat{F}^\dagger$

HONKONEN, LAUSE 6.4 :

$$\hat{F}(-i\hat{E}_x)(k) = k \tilde{\psi}(k)$$

$$\text{ELI } \hat{F}(\hat{P}\psi)(x) = k \tilde{\psi}(k)$$

$$= (\hat{X}\tilde{\psi})(k) = \hat{X} \hat{F}\psi(x)$$

KERTOMALLA \hat{F}^{-1} :LLÄ VASEMMALTA

$$\Rightarrow (\hat{P}\psi)(x) = (\hat{F}^{-1}\hat{X}\hat{F})\psi(x)$$

KAIKILLE $\psi \in L^2$, ELI

$$\hat{P} = \hat{F}^{-1}\hat{X}\hat{F}$$

\hat{X} :N SPEKTRAALIESITYKSESTÄ
SAADAN \hat{P} :N SPEKTRAALIESITYS

$$\hat{P} = \int_{-\infty}^{\infty} k d(\hat{F}^{-1}\hat{E}_k\hat{F})$$

(INTEGROIMISMUUTTUAU NIKEÄ VAHDETTU $x \rightarrow k$)

$$\tilde{\psi}(k) = \langle k | \psi \rangle = \int_{-\infty}^{\infty} dx \langle k | x \rangle \psi(x)$$

$$\stackrel{\text{OHJE}}{\sim} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \psi(x)$$

KONSISTENSSI VAATII SIIS

$$\langle k | x \rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$$

JÄ

$$\varphi_k(x) \equiv \langle x | k \rangle = \langle k | x \rangle^*$$

$$= \frac{1}{\sqrt{2\pi}} e^{ikx}$$

"IMPULSIOPERAATTORIN OHINAIS.

FUNKTIO" ($\varphi \in L^2(\mathbb{R})$!)

$$\langle \varphi | \hat{p} | \psi \rangle = \int_{-\infty}^{\infty} k dx \langle \varphi | \hat{x}^{-1} \hat{p}^{-1} \hat{x} | \psi \rangle$$

$$\text{NYT } \langle \varphi | \hat{x}^{-1} \hat{p}^{-1} \hat{x} | \psi \rangle \stackrel{!}{=} \langle \varphi | \hat{x}^{-1} \hat{p}^{-1} \hat{x} | \psi \rangle$$

$$= \langle \hat{x} \varphi | \hat{p}^{-1} | \hat{x} \psi \rangle = \int_{-\infty}^{\infty} dx k' \tilde{\varphi}(k')^* \tilde{\psi}(k)$$

$$\Rightarrow \langle \varphi | \hat{p} | \psi \rangle = \int_{-\infty}^{\infty} dx k \tilde{\varphi}^*(k) \tilde{\psi}(k)$$

$$= \int_{-\infty}^{\infty} dx k \tilde{\varphi}^*(k) k \tilde{\psi}(k) \stackrel{!}{=} \int_{-\infty}^{\infty} dx k \tilde{\varphi}^*(k) \tilde{\psi}(k)$$

$$= \int_{-\infty}^{\infty} dx \varphi(x)^* \left(-i \frac{d}{dx} \psi \right) \quad \text{O.K. !}$$

DIRACIN MERKINTÄTAVASSA

OSTAN KÄYTTÖÖN "VEKTORBITA"

$$|k\rangle \text{ s.e. } \langle k | k' \rangle = \delta(k - k')$$

$$\tilde{\varphi}(k) = \langle k | \varphi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \varphi(x)$$

$$\text{NYT } |\varphi\rangle = i\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dx k |k\rangle \langle k | \psi \rangle$$

$$= \int_{-\infty}^{\infty} dx k |k\rangle \varphi(k)$$

TRIGONOMETRIAN

$$\langle \cos | \cos \rangle = \langle \cos | \cos \rangle$$

$$\int_{-\pi}^{\pi} \cos(x) \cos(x) dx$$

$$= \int_{-\pi}^{\pi} \cos^2(x) dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2x)}{2} dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos(2x)) dx = \frac{1}{2} [x + \frac{\sin(2x)}{2}]_{-\pi}^{\pi}$$

$$= \frac{1}{2} (\pi - (-\pi) + \frac{\sin(2\pi)}{2} - \frac{\sin(-2\pi)}{2}) = \pi$$

$$\langle \cos | \cos \rangle = \pi$$

TRIGONOMETRIAN

$$\int_{-\pi}^{\pi} \cos(x) \sin(x) dx = 0$$

TRIGONOMETRIAN

$$\langle \sin | \sin \rangle = \int_{-\pi}^{\pi} \sin^2(x) dx$$

$$= \int_{-\pi}^{\pi} \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} [x - \frac{\sin(2x)}{2}]_{-\pi}^{\pi}$$

TRIGONOMETRIAN!

LEGENDRE

$$-\frac{d}{dx} \left[(1-x^2) \frac{dP_n}{dx} \right] = \lambda(\lambda+1) P_n(x)$$

$$a = -1, b = 1 \quad P_n(x) = 1-x^{2n} \quad \text{SINGULÄRITÄTEN}$$

$$q(x) = 0$$

$$w(x) = 1$$

$$\lambda = \lambda(\lambda+1)$$

LEGENDREN LITTOFUNKTIOT

$$-\frac{d}{dx} \left[(1-x^2) \frac{dP_n}{dx} \right] + \frac{m^2}{1-x^2} P_n(x) = \lambda(\lambda+1) P_n(x)$$

$$a = -1, b = 1 \quad P_n(x) = 1-x^{2n}, \quad q(x) = \frac{m^2}{1-x^2}, \quad w(x) = 1$$

$$\lambda = \lambda(\lambda+1)$$

BESSEL $R > 0$ $J_\nu(x)$ JA $Y_\nu(x)$ NOLLAKOIKAT

$$x = \frac{R}{\xi} \quad 0 \leq \xi \leq 1 \quad a = 0, b = 1$$

$$-\frac{d}{d\xi} \left[\xi \frac{dJ_\nu(\xi R)}{d\xi} \right] + \frac{\nu^2}{\xi} J_\nu(\xi R) = R^2 \xi J_\nu(\xi R)$$

$$P(\xi) = \frac{\nu^2}{\xi}, \quad q(\xi) = \frac{\nu^2}{\xi}, \quad w(\xi) = \xi, \quad \lambda = R^2$$

LACUERRE

$$-\frac{d}{dx} \left[x e^{-x} \frac{dL_n}{dx} \right] = n e^{-x} L_n(x) \quad 0 \leq x < \infty$$

$$P(x) = x e^{-x} \quad q(x) = 0, \quad w(x) = e^{-x}, \quad \lambda = n$$