FYSIIKAN MATEMAATTISET MENETELMÄT III. Harjoitus 7 sl 2005.

Palautetaan ke 2.11., käsitellään ma 7.11.
Huom.! Ke 26.10. ei luentoja. Matkan vuoksi luennot 2.11., 9.11. siirretään pidettäviksi to 10.11. klo 12-14 ja pe 11.11. klo 14-16, molemmat salissa D112.
Tehtävien teksti kokeiluluontoisesti vain englanniksi tällä kertaa.

English text:
Return your solutions by Wed Nov. 2^{nd}, exercise session on Mon, Nov. 7^{th}.
Note: No lectures on Wed Oct. 26^{th}. Due to travels, I have to move the lectures on November 2^{nd} and 9^{th} to: Thursday, November 10 12-14 hrs and Friday, November 11 14-16 hrs, both times in D112.
As an experiment, the text of the problems is given in English only this time.

1. A pseudo-Euclidean space is a real vector space with an inner product $u \cdot v$, where instead of positivity ($u \cdot u \geq 0$ for all u) one requires only non-degeneracy: If $u \cdot v = 0$ for all v then $u = 0$.
a) Show that the matrix formed by the components of the metric tensor $g(u,v) = u \cdot v$ in any basis is non-singular.
b) Show that we can always find a basis such that $g_{ij} = e_i \cdot e_j = \eta_i \delta_{ij}$, where $\eta_i = \pm 1$. We call such a basis an orthonormal (ON) basis.
c) Prove Sylvester's theorem which states that the number of + and – signs among the η_i is the same for any ON basis.

2. Let V be a pseudo-Euclidean vector space with inner product $u \cdot v$ and an ON basis $\{e_i\}$: $g_{ij} = e_i \cdot e_j = \pm\delta_{ij}$. The *Clifford algebra* C_g associated with V is the associative algebra generated by (i.e. consists of all linear combinations of all products of) the elements $1, \gamma_1, ..., \gamma_n$ ($n = \dim V$) with the product rules

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2 g_{ij} 1,$$
$$1\gamma_i = \gamma_i 1 = \gamma_i .$$

a) Show that $n = 1$, $g_{11} = -1$ gives rise to the complex numbers and $n = 2$, $g_{ij} = -\delta_{ij}$ the quaternions.
b) Find a basis of C_g regarded as a vector space. What is the dimension of C_g for a given n?
c) Find a matrix representation of C_g for $n = 4$, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ (the Dirac algebra).

3. a)If α is an r-form and β an s-form, show that $\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$.
b) Show that the vectors $u_1, ..., u_r$ are linearly dependent if and only if $u_1 \wedge u_2 \wedge ... \wedge u_r = 0$.
c) Let $\dim V = 4$ and $\{e_1, e_2, e_3, e_4\}$ be a basis of V. Let A be the 2-vector

$$A = e_2 \wedge e_1 + a e_1 \wedge e_3 + e_2 \wedge e_3 + c e_1 \wedge e_4 + b e_2 \wedge e_4 .$$

Show that the equation $A \wedge u = 0$ has a nontrivial solution $u \neq 0$ if and only if $c = ab$. In this case find vectors u,v such that $A = u \wedge v$.

4. Let $\{e_1, e_2, e_3\}$ be a basis of V and $\{e'_1, e'_2, e'_3\}$ a second basis given by

$$e'_1 = e_1 - e_2, e'_2 = e_3, e'_3 = e_1 + e_2 .$$

a) Find the new dual basis $\{\varepsilon'^i\}$ in terms of the old dual basis $\{\varepsilon^i\}$.
b) What are the components of the tensors

$$T = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_3 \text{ and}$$
$$S = e_1 \otimes \varepsilon^1 + 3e_1 \otimes \varepsilon^3 - 2e_2 \otimes \varepsilon^3 - e_3 \otimes \varepsilon^1 + 4e_3 \otimes \varepsilon^2$$

in terms of the basis $\{e_1, e_2, e_3\}$ and its dual basis?

c) What are the components of these tensors in terms of the basis $\{e'_1, e'_2, e'_3\}$ and its dual basis?

5. a)Show that $i_u(i_v\alpha) = -i_v(i_u\alpha)$ and hence that $i_u^2 = 0$.

b) Show $i_u(\alpha \wedge \beta) = (i_u\alpha) \wedge \beta + (-1)^r \alpha \wedge (i_u\beta)$ if α is an r-form.

c) In 3 dimensions, show

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta_j^m \delta_k^n - \delta_k^m \delta_j^n,$$
$$\varepsilon_{ijk}\varepsilon^{ijn} = 2\delta_k^n,$$
$$\varepsilon_{ijk}\varepsilon^{ijk} = 6,$$

and use the first identity to prove the triple cross product rule

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.$$