MUUTELLUT BESSELIN FUNKTIOT

$I_\nu(z)$, $K_\nu(z)$

TOTEUTTAVAT

TÄMÄN TERMIN ETUMERKKI VAIHTUNUT

$$\frac{d^2f}{dz^2} + \frac{1}{z}\frac{df}{dz} - \left(1 + \frac{\nu^2}{z^2}\right) f = 0$$

$z \to iz$ BESSELIN YHTÄLÖSSÄ $\Rightarrow Z_\nu(iz)$

ON RATKAISU, $Z = J, Y, H^{(1,2)}$

$$I_\nu(z) = e^{-\frac{i\nu\pi}{2}} J_\nu(iz) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$

JOTTA $I_\nu(x)$ OLISI REAALINEN, KUN $x \in \mathbb{R}$

ON $z=0$:SSA SÄÄNNÖLLINEN RATKAISU $(\nu \geq 0)$

JOTTA $K_\nu(x) \in \mathbb{R}$ KUN $x \in \mathbb{R}$

$$K_\nu(z) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz) = \frac{\pi}{2} i^{\nu+1}\left(J_\nu(iz) + i Y_\nu(iz)\right)$$

$$= \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

"MACDONALDIN FUNKTIO" (HECTOR MACDONALD 1865-1935)

$z \to 0$ $\quad K_0(z) = -\log\left(\frac{z}{2}\right) - \gamma + \dots$

$$K_\nu(z) = \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} + \dots$$

HYÖDYLLINEN INTEGRAALIESITYS:

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-t-\frac{z^2}{4t}}$$

PALAUTUSKAAVOJA

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$

$\Gamma(\nu+k+1) = (\nu+k)\Gamma(\nu+k)$

$$\Rightarrow \frac{d}{dz}\left(z^\nu J_\nu(z)\right) = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k z^{2\nu+2k}}{2^{\nu+2k}\,k!\,\Gamma(\nu+k+1)} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k)} \left(\frac{z}{2}\right)^{\nu+2k-1} \cdot z^\nu = z^\nu J_{\nu-1}(z)$$

$$\Rightarrow \underline{\frac{1}{z}\frac{d}{dz}\left(z^\nu J_\nu(z)\right) = z^{\nu-1} J_{\nu-1}(z)} \qquad (1)$$

TOISTETAAN m KERTAA $\Rightarrow$

$$\underline{\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^\nu J_\nu(z)\right) = z^{\nu-m} J_{\nu-m}(z)} \qquad (2)$$

$m = 1, 2, \dots$

VASTAAVASTI

$$\frac{d}{dz}\left(z^{-\nu} J_\nu(z)\right) = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{\nu+2k}\,k!\,\Gamma(\nu+k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-1}}{2^{\nu+2k-1}(k-1)!\,\Gamma(\nu+k-1)} \overset{q=k-1}{=}$$

$$= -\sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+1}}{2^{\nu+2q+1}\,q!\,\Gamma(\nu+q+2)} \cdot z^{-\nu} \cdot z^{+\nu}$$

$$= -z^{-\nu}\sum_{q=0}^{\infty} \frac{(-1)^q}{q!\,\Gamma(\nu+1+q+1)} \left(\frac{z}{2}\right)^{2q+\nu+1} = -z^{-\nu} J_{\nu+1}(z)$$

$$\Rightarrow \underline{\frac{1}{z}\frac{d}{dz}\left(z^{-\nu} J_\nu(z)\right) = -z^{-(\nu+1)} J_{\nu+1}(z)} \qquad (3)$$

TOISTETAAN m KERTAA $\Rightarrow$

$$\underline{\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^{-\nu} J_\nu(z)\right) = (-1)^m z^{-(\nu+m)} J_{\nu+m}(z)} \qquad (4)$$

(1) SANOO $z^{\nu} J_\nu' + \nu z^{\nu-1} J_\nu = z^{\nu} J_{\nu-1}$

ELI $zJ_\nu' + \nu J_\nu - z J_{\nu-1} = 0$ (5)

(3) SANOO $z^{-\nu} J_\nu' - \nu z^{-\nu-1} J_\nu = - z^{-\nu} J_{\nu+1}$

ELI $zJ_\nu' - \nu J_\nu + z J_{\nu+1} = 0$ (6)

(5) + (6) $\Rightarrow$ $2zJ_\nu' + z\left(J_{\nu+1} - J_{\nu-1}\right) = 0$ ELI

$$\boxed{J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z)}$$ (PAINOVIRHE CROUSTRÖKISSÄ)

(5) − (6) $\Rightarrow$ $2\nu J_\nu - z\left(J_{\nu+1} + J_{\nu-1}\right) = 0$ ELI

$$\boxed{J_{\nu+1}(z) + J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z)}$$

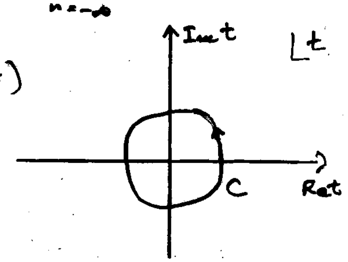
MUODOSTAJAFUNKTIO

$$G(z,t) \equiv e^{\frac{z}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

TODISTUS:

KIRJOITAMME $e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = G(z,t) = \sum_{n=-\infty}^{\infty} a_n(z) t^n$

$$\Rightarrow a_n(z) = \frac{1}{2\pi i} \oint_C dt\, t^{-n-1} G(z,t)$$



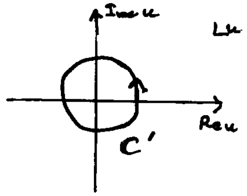
C : ORIGOKESKEINEN YMPYRÄ VASTAPÄIVÄÄN

$$a_n(z) = \frac{1}{2\pi i} \oint_C \frac{dt}{t^{n+1}}\, e^{\frac{z}{2}\left(t - \frac{1}{t}\right)}$$

MUUTTUJAN VAIHDOS : $t \to u = \frac{zt}{2}$ (OL. $z \neq 0$)

$$\frac{dt}{t^{n+1}} = \left(\frac{z}{2}\right)^n \frac{du}{u^{n+1}} \qquad |u| = \left|\frac{z}{2}\right| |t|$$

$$\Rightarrow a_n(z) = \left(\frac{z}{2}\right)^n \frac{1}{2\pi i} \oint_{C'} \frac{du}{u^{n+1}}\, e^{u} \underbrace{e^{-z^2/4u}}_{\sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\left(\frac{z}{2}\right)^{2r} u^{-r}}$$



$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{z}{2}\right)^{n+2r} \frac{1}{2\pi i} \oint_{C'} du\, u^{-(r+n+1)} e^{u}$$

NYT $\oint_{C'} du\, u^{-(r+n+1)} e^{u} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\frac{1}{2\pi i} \oint_{C'} du\, u^{-r-n-1+k}}_{\delta_{k,r+n}}$

$$= \frac{1}{(r+n)!} = \frac{1}{\Gamma(r+n+1)}$$

SIIS

$$a_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \left(\frac{z}{2}\right)^{2r+n} = J_n(z)$$

$n = 0, \pm 1, \pm 2, \ldots$ □

$z = 0$: $G(0,t) = 1 = \sum_{n=-\infty}^{\infty} a_n(0) t^n$

$\Rightarrow$ $a_0(0) = 1$, $a_n(0) = 0$ $n \neq 0$ } PÄTEE MYÖS $J_n(0)$:LLE

SEURAUKSIA:

1) GEN. FUNKTIO: $G(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$

JOS $t = e^{i\varphi}$ $\quad t - \frac{1}{t} = e^{i\varphi} - e^{-i\varphi} = 2i \sin\varphi$

$$\Rightarrow \quad \underbrace{e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\varphi}} \qquad (F)$$

(F) ON FOURIER'N SARJA, KERROINTEN KAAVASTA

$$\Rightarrow \boxed{J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \; e^{i(x\sin\varphi - n\varphi)}} \quad n \in \mathbb{Z}$$

↑ INTEGRAALI SAA OLLA YLI MIKÄ TAHANSA 2π-PITUISEN VÄLIN

KUN $x \in \mathbb{R}$, $\operatorname{Im} J_n(x) = 0$

$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \sin(x\sin\varphi - \varphi) = \int_{-\pi}^{\pi} d\varphi \cdot$ PARITON FUNKTIO

$= 0$ OK.

$$\Rightarrow J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \cos(x\sin\varphi - n\varphi)$$

$$= \underbrace{\frac{1}{\pi} \int_{0}^{\pi} d\varphi \cos(x\sin\varphi - n\varphi)}$$

TOINEN EKVIVALENTTI MUOTO

2) KUN $x \in \mathbb{R}$ KAAVAN (F) REAALIOSA ON

$$\cos(x\sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \cos n\varphi = J_0(x) + \sum_{n=1}^{\infty} (J_n + \underbrace{J_{-n}}_{(-1)^n J_n}) \cos n\varphi$$

$$= \underbrace{J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(2n\varphi)}$$

IMAGINAARIOSA TAAS ANTAA

$$\sin(x\sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \sin n\varphi$$

$$= \sum_{n=1}^{\infty} \left(J_n(x) - (-1)^n J_n(x)\right) \sin n\varphi$$

$$= \underbrace{2 \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1)\varphi}$$

3) "YHTEENLASKUTEOREEMA"

$$G(x+y,t) = G(x,t)G(y,t) = \sum_{p=-\infty}^{\infty} J_p(x) t^p \sum_{q=-\infty}^{\infty} J_q(y) t^q$$

$$\sum_{n=-\infty}^{\infty} J_n(x+y) t^n \;=\; \sum_{n=-\infty}^{\infty} \left(\sum_{p=-\infty}^{\infty} J_p(x) J_{n-p}(y) \right) t^n$$

$$\Rightarrow \boxed{J_n(x+y) = \sum_{p=-\infty}^{\infty} J_p(x) J_{n-p}(y)}$$

ERIKOISESTI JOS $n=0$ JA $x = -y$

$$J_0(0) = 1 = \sum_{p=-\infty}^{\infty} J_p(x) \underbrace{J_{-p}(-x)}_{(-1)^p J_p(-x) = J_p(x)} = \sum_{p=-\infty}^{\infty} J_p^2(x)$$

$$\Leftrightarrow \boxed{J_0^2(x) + 2\sum_{p=1}^{\infty} J_p^2(x) = 1}$$

$$\Rightarrow |J_0(x)| \leq 1 \; ; \; |J_p(x)| \leq \frac{1}{\sqrt{2}} \quad p = 1, 2, \ldots$$

INTEGRAALIESITYKSIÄ

$$G(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

JOHDETTIIN:

$$J_n(x) = \frac{1}{2\pi i} \oint \frac{dt}{t^{n+1}}\, e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \qquad n = 0, \pm 1, \pm 2, \ldots$$

"SCHLÄFLIN INTEGRAALI" (LUDWIG SCHLÄFLI 1814-1895)

POISSONIN INTEGRAALIESITYS: (SIMÉON-DENIS POISSON 1781-1840)

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} du\, e^{iuz} (1-u^2)^{\nu-\frac{1}{2}}$$

$$\mathrm{Re}\,\nu > -\frac{1}{2}$$

JOHTO: KATSO CRONSTRÖM

KÄYTTÄYTYMINEN PIENILLÄ JA SUURILLA ARGUMENTIN ARVOILLA

SARJASTA $x \to 0$:

$$J_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \left(1 + O(x^2)\right)$$

$$\nu \neq -1, -2, \ldots$$

(TÄMÄ TULOS KÄYTETTIIN JO)

$$J_{-n}(x) = (-1)^n J_n(x) \xrightarrow[x \to 0]{} \frac{(-1)^n}{n!} \left(\frac{x}{2}\right)^n \left(1 + O(x^2)\right)$$

$x \in \mathbb{R}$, $|x| \to \infty$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \left(\nu+\tfrac{1}{2}\right)\tfrac{\pi}{2}\right) + O\left(\tfrac{1}{x}\right) \right)$$

TODISTUS: KATSO KIRJALLISUUS (ESIM. CRONSTRÖM PYKÄLÄ 4.2.9)

SEURAUS: FUNKTIOLLA $J_\nu(x)$ ON ÄÄRETTÖMÄN MONTA NOLLAKOHTAA REAALIAKSELILLA

ORTOGONAALITEETTI

FUNKTIOT $u(x) = J_\nu(ax)$ JA $v(x) = J_\nu(bx)$

TOTEUTTAVAT

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{\nu^2}{x^2}\right)u = 0 \quad \Big| \cdot xv$$

$$v'' + \frac{1}{x}v' + \left(b^2 - \frac{\nu^2}{x^2}\right)v = 0 \quad \Big| \cdot xu$$

VÄHENNETÄÄN TOISISTAAN

$$\Rightarrow \underbrace{x(vu'' - uv'') + (vu' - uv')}_{= \frac{d}{dx}\left[x(u'v - uv')\right]} + (a^2 - b^2)\,xuv = 0$$

INTEGROIDAAN $\int_0^R dx$:

$$\Rightarrow (b^2 - a^2)\int_0^R dx\, x\, J_\nu(ax)\, J_\nu(bx) =$$

$$= \Big/_0^R x\left[a J_\nu'(ax) J_\nu(bx) - b J_\nu(ax) J_\nu'(bx)\right] \qquad (1)$$

PALAUTUSKAAVOISTA SAADAAN (OL. ENSIN $\nu \neq 0$)

$$J_\nu(\alpha x) = \frac{\alpha x}{2\nu}\left(J_{\nu-1}(\alpha x) + J_{\nu+1}(\alpha x)\right) \qquad \alpha = a \text{ TAI } b$$

$$J_\nu'(\alpha x) = \frac{1}{2}\left(J_{\nu-1}(\alpha x) - J_{\nu+1}(\alpha x)\right) \qquad \text{— " —}$$

$$\Rightarrow x\left[a J_\nu'(ax) J_\nu(bx) - b J_\nu(ax) J_\nu'(bx)\right]$$

$$= \frac{abx^2}{2\nu}\left[J_{\nu-1}(ax) J_{\nu+1}(bx) - J_{\nu+1}(ax) J_{\nu-1}(bx)\right]$$

$$\approx \text{vakio} \cdot x^{2\nu+2} \to 0 \quad \text{KUN } x \to 0 \quad \text{JOS } \nu > -1$$

$\nu = 0$:

$$J_0(\alpha x) = 1 + k(\alpha x)^2 + \dots$$

$$J_0'(\alpha x) = 2k\alpha x + \dots$$

$$\Rightarrow x\left[a J_0'(ax) J_0(bx) - b J_0(ax) J_0'(bx)\right] = \text{vakio} \cdot x^2 \underset{x \to 0}{\to} 0$$

VALITAAN NYT a, b S.E. SIJOITUSTERMI $= 0$ MYÖS YLÄRAJALLA, ESIM.

$$J_\nu(aR) = J_\nu(bR) = 0$$

OLKOON x_{0i} $J_\nu(x)$:N i:S NOLLAKOHTA KUN $x > 0$ (NOLLAKOHTIA OLI ∞:N MONTA)

VAL. $a = \frac{x_{0i}}{R}$, $b = \frac{x_{0j}}{R}$

$$\Rightarrow \frac{1}{R^2}(x_{0j}^2 - x_{0i}^2)\int_0^R dx\, x\, J_\nu\!\left(\frac{x_{0i}}{R}x\right) J_\nu\!\left(\frac{x_{0j}}{R}x\right) = 0$$

JOS $x_{0i} \neq x_{0j}$

$$\Rightarrow \int_0^R dx\, x\, J_\nu\!\left(\frac{x_{0i}}{R}x\right) J_\nu\!\left(\frac{x_{0j}}{R}x\right) = 0 \qquad \underline{\nu > -1}$$

MITÄ JOS $x_{0i} = x_{0j}$?

KAAVASSA (1) VALITAAN $a = \frac{x_{0i}}{R}$, $b = \frac{x_{0i}}{R} + \varepsilon$

$$b^2 - a^2 = (b+a)(b-a) = \frac{2x_{0i}}{R}\cdot\varepsilon + O(\varepsilon^2)$$

$$\Rightarrow \int_0^R dx\, x\left[J_\nu\!\left(\frac{x_{0i}}{R}x\right)\right]^2 = \lim_{\varepsilon\to 0}\frac{1}{\frac{2x_{0i}}{R}\cdot\varepsilon}\cdot R\Big[\frac{x_{0i}}{R} J_\nu'(x_{0i}) J_\nu(x_{0i} + \varepsilon R)$$

$$- \Big(\frac{x_{0i}}{R} + \varepsilon\Big)\underbrace{J_\nu(x_{0i})}_{=0} J_\nu'(x_{0i} + \varepsilon R)\Big]$$

$$= \frac{R}{2} J_\nu'(x_{0i}) \cdot \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} J_\nu(x_{0i} + \varepsilon R) = \frac{R^2}{2}\left[J_\nu'(x_{0i})\right]^2$$

PALAUTUSKAAVAT :

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2 J_\nu'(x)$$

LASKEMALLA YHTEEN $\Rightarrow$ $J_\nu'(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x)$

VÄHENTÄMÄLLÄ $\Rightarrow$ $J_\nu'(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x)$

$$\Rightarrow \quad J_\nu'(x_{0i}) = J_{\nu-1}(x_{0i}) = - J_{\nu+1}(x_{0i})$$

$$\Rightarrow \boxed{\int_0^R dx\, x\, J_\nu\left(\frac{x_{0i}}{R}x\right) J_\nu\left(\frac{x_{0j}}{R}x\right) = \frac{R^2}{2}\left[J_{\nu\pm1}(x_{0i})\right]^2 \delta_{ij}} \quad \underline{\underline{\nu > -1}}$$

PALLOBESSELIT

HELMHOLTZIN YHTÄLÖN $(\nabla^2 + k^2)u = 0$ SEPAROINTI PALLOKOORDINAATEISSA JOHTAA RADIAALISEEN YHTÄLÖÖN

$$r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} + \left[k^2r^2 - \ell(\ell+1)\right]R = 0$$

SIIRRYTÄÄN MUUTTUJAAN $x = kr$

$$x^2 \frac{d^2R}{dx^2} + 2x \frac{dR}{dx} + \left[x^2 - \ell(\ell+1)\right]R = 0$$

YRITE: $R(x) = \frac{Z(x)}{\sqrt{x}}$

$$\Rightarrow x^2R'' + 2xR' + \left[x^2 - \ell(\ell+1)\right]R =$$

$$= \frac{1}{\sqrt{x}}\left(x^2 Z'' + x Z' + \left(x^2 - (\ell+\tfrac{1}{2})^2\right) Z \right) = 0$$

$\Rightarrow$ Z SYLINTERIFUNKTIO INDEKSILLÄ $\ell + \frac{1}{2}$

MÄÄRITELLÄÄN

PALLOBESSEL $\quad j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$

PALLONEUMANN $\quad n_n(x) = \sqrt{\frac{\pi}{2x}}\, N_{n+\frac{1}{2}}(x)$

$$= (-1)^{n+1}\sqrt{\frac{\pi}{2x}}\, J_{-n-\frac{1}{2}}(x)$$

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\frac{1}{\Gamma(\nu+s+1)}\left(\frac{x}{2}\right)^{\nu+2s}$$

ANTAA

$$J_{\frac{1}{2}}(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{1}{\Gamma(s+\frac{3}{2})}\left(\frac{x}{2}\right)^{2s}\sqrt{\frac{x}{2}}$$

NYT $\quad \Gamma(s+\frac{3}{2}) = (s+\frac{1}{2})\Gamma(s+\frac{1}{2}) = \dots = (s+\frac{1}{2})(s-\frac{1}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

$$= \frac{(2s+1)!!}{2^{s+1}}\sqrt{\pi}$$

$(2s+1)!! = 1\cdot 3\cdot 5 \cdots (2s+1)$ $\qquad 2^s s!\,(2s+1)!! = (2s+1)!$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{2^{s+1}}{(2s+1)!!\sqrt{\pi}}\frac{x^{2s}}{2^{2s}} =$$

$$= \sqrt{\frac{2x}{\pi}}\sum_{s=0}^{\infty}\frac{(-1)^s x^{(2s+1)}}{(2s+1)!}\frac{1}{x} = \sqrt{\frac{2x}{\pi}}\frac{\sin x}{x}$$

$$\Rightarrow j_0(x) = \frac{\sin x}{x}$$

PALAUTUSKAAVA $\quad J_{\nu+1} = \frac{\nu}{x}J_\nu - J_\nu' = -x^\nu\frac{d}{dx}(x^{-\nu}J_\nu)$

$$\Rightarrow x^{-(\nu+1)}J_{\nu+1} = -\frac{1}{x}\frac{d}{dx}(x^{-\nu}J_\nu) = \left(-\frac{1}{x}\frac{d}{dx}\right)^2\left(x^{-(\nu-1)}J_{\nu-1}\right) = \cdots$$

$$\Rightarrow x^{-(n+\frac{1}{2})}J_{n+\frac{1}{2}}(x) = (-1)^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{1}{\sqrt{x}}J_{\frac{1}{2}}(x)\right)$$

$$\Rightarrow j_n(x) = x^n(-1)^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}$$

$\Rightarrow j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$, $\; j_2(x) = \left(\frac{3}{x^3}-\frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$ jne.

VASTAAVASTI $\quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2x}{\pi}}\,\frac{\cos x}{x}$,

JOSTA $\quad n_0(x) = -\frac{\cos x}{x}$

PALAUTUSKAAVASTA $\quad J_{\nu-1} = \frac{\nu}{x}J_\nu + J_\nu'$

$$= x^{-\nu}\frac{d}{dx}(x^\nu J_\nu)$$

SEURAA $\quad x^{\nu-1}J_{\nu-1}(x) = \frac{1}{x}\frac{d}{dx}(x^\nu J_\nu)$

$$\Rightarrow J_{-n-\frac{1}{2}}(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{1}{\sqrt{x}}J_{-\frac{1}{2}}(x)\right)$$

$$\Rightarrow n_n(x) = (-1)^n\left(\frac{1}{x}\frac{d}{dx}\right)^n n_0(x)$$

$$= (-1)^{n+1}\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x}$$

$$\Rightarrow n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$n_2(x) = -\left(\frac{3}{x^3}-\frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

RAJA-ARVOJA:

$x \to 0 \qquad j_n(x) \simeq \frac{x^n}{(2n+1)!!}$

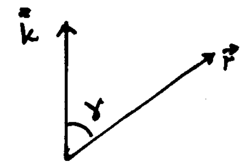
$$n_n(x) \simeq -(2n-1)!!\, x^{-n-1} \qquad n \geq 1$$

$$n_0(x) \simeq -\frac{1}{x}$$

$|x| \to \infty \qquad j_n(x) \to \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right)$

$$n_n(x) \to -\frac{1}{x}\cos\left(x-\frac{n\pi}{2}\right)$$

TASOAALLON PALLOAALTOKEHITELMÄ



$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\gamma}$$

$$= \sum_{l=0}^{\infty}(2l+1)\, f_l(kr)\, P_l(\cos\gamma)$$

$$f_l(kr) = \frac{1}{2}\int_{-1}^{1} d\cos\gamma\; e^{ikr\cos\gamma}\, P_l(\cos\gamma)$$

OSOITAMME, ETTÄ $f_l(kr) \propto j_l(kr)$:

POISSONIN INTEGRAALIESITYS (CRONSTRÖM §4.2.6)

$$\Rightarrow \sqrt{\frac{2\pi}{x}}\, J_{l+\frac{1}{2}}(x) = \frac{x^l}{2^l l!}\int_{-1}^{1} du\, e^{iux}(1-u^2)^l$$

OSITTAISINTEGROIDAAN:

$$\int_{-1}^{1} du\, e^{iux}(1-u^2)^l = \frac{i}{x}\int_{-1}^{1} du\, e^{iux}\frac{d}{du}(1-u^2)^l = \cdots$$

$$= \left(\frac{i}{x}\right)^l \int_{-1}^{1} du\, e^{iux}\frac{d^l}{du^l}(1-u^2)^l$$

$$\Rightarrow \underbrace{\sqrt{\frac{2\pi}{x}}\, J_{l+\frac{1}{2}}}_{=\,2 j_l(x)} = (-i)^l \int_{-1}^{1} du\, e^{iux}\underbrace{\frac{1}{2^l l!}\frac{d^l}{du^l}(u^2-1)^l}_{P_l(u)\ \text{(RODRIGUES)}}$$

$$\Rightarrow f_l(kr) = i^l\, j_l(kr)$$

$$\Rightarrow e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty}(2l+1)\, i^l\, j_l(kr)\, P_l(\cos\gamma)$$

$$= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l\, j_l(kr)\, Y_{lm}(\theta_k,\varphi_k)\, Y^*_{lm}(\theta_r,\varphi_r)$$

YHTEENLASKULAUSE

MISSÄ MERKITTIIN $\vec{k} = k(\sin\theta_k\cos\varphi_k,\ \sin\theta_k\sin\varphi_k,\ \cos\theta_k)$

$\vec{r} = r(\sin\theta_r\cos\varphi_r,\ \sin\theta_r\sin\varphi_r,\ \cos\theta_r)$

LAGUERREN FUNKTIOT

(EDMOND LAGUERRE 1834-1886)

LAGUERREN YHTÄLÖ :

$$z \frac{d^2 f}{dz^2} + (1-z)\frac{df}{dz} + \underset{\uparrow \text{PARAMETRI}}{\mu} f(z) = 0$$

$z=0$ HEIKKO ERIKOISPISTE

POTENSSISARJAYRITE : $f(z) = \sum_{k=0}^{\infty} b_k z^{k+r}$

$z f'' = b_0 r(r-1) z^{r-1} + \dots$

$(1-z) f' = b_0 r z^{r-1} + \dots$

$\mu f = \mu b_0 z^r + \dots$

KARAKTERISTINEN YHTÄLÖ: $r(r-1) + r = r^2 = 0$

$r=0$ KAKSOISJUURI $\Rightarrow$ SAADAAN $z=0$:SSA

SÄÄNNÖLLINEN RATKAISU $f(z) = \sum_{k=0}^{\infty} b_k z^k = \mathcal{L}_\mu(z)$

JA TOINEN RATKAISU $\propto \log z$, $z \to 0$

↑ ETSITÄÄN TÄTÄ "LAGUERREN FUNKTIO" (TIETYLLÄ $\mathcal{L}_\mu(0)$:N VALINNALLA)

$r=0 \Rightarrow$ PALAUTUSKAAVA

$$b_{k+1} = \frac{(k-\mu)}{(k+1)^2} b_k$$

RATKAISU

$$b_k = \frac{b_k}{b_{k-1}} \frac{b_{k-1}}{b_{k-2}} \dots \frac{b_1}{b_0} b_0$$

$$= \frac{(k-1-\mu)}{k^2} \frac{(k-2-\mu)}{(k-1)^2} \dots \frac{(-\mu)}{1^2} b_0$$

$$= \frac{(-\mu)_k}{(k!)^2} b_0$$

KÄYTTÄYTYMINEN ISOILLA $|z|$:

$$\frac{b_m}{b_{m-1}} = \frac{m-1-\mu}{m^2} \underset{m\to\infty}{\longrightarrow} \frac{1}{m}$$

$$b_m \propto \frac{1}{m!} \qquad m \to \infty$$

$$\Rightarrow \mathcal{L}_\mu(z) \sim e^{+az} \qquad \underset{z \to +\infty}{(a>0)}$$

POIKKEUS : JOS $\mu = n$, KOKONAISLUKU

$(-n)_k = 0 \quad k > n \qquad \Rightarrow \mathcal{L}_n(z)$ ON <u>POLYNOMI</u> ASTETTA n

$$= (-1)^k \frac{n!}{(n-k)!} \quad \left((-n)_k = (k-1-n)(k-2-n)\cdots(-n) = (-1)^k n \cdot (n-1) \cdots (n-k+1) \right)$$

$$\Rightarrow b_k = n! \frac{(-1)^k}{(k!)^2 (n-k)!} b_0$$

KUN VALITAAN $b_0 = n!$ SAADAAN <u>LAGUERREN POLYNOMIT</u>

$$\mathcal{L}_n(z) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^k}{(k!)^2 (n-k)!} z^k$$

$$= (-1)^n z^n + \dots$$

ESIM. $\mathcal{L}_0(z) = 1$

$\mathcal{L}_1(z) = 1 - z$

$\mathcal{L}_2(z) = 2 - 4z + z^2$

$\mathcal{L}_3(z) = 6 - 18z + 9z^2 - z^3$ JNE

HUOM! MUITAKIN NORMITUSSOPIMUKSIA LÖYTYY !

LAGUERREN LIITTOPOLYNOMIT

$$\mathcal{L}_n^{(p)} \equiv \frac{d^p}{dz^p}\mathcal{L}_n(z) \qquad p = 0, 1, \ldots, n$$

$$\mathcal{L}_n^{(0)}(z) = \mathcal{L}_n(z) \qquad \mathcal{L}_n^{(n)}(z) = (-1)^n n! \quad \text{VAKIO}$$

$$\mathcal{L}_n^{(p)}(z) = \sum_{k=p}^{n} \frac{(-1)^k (n!)^2}{k!(n-k)!(k-p)!} z^{k-p}$$

TOTEUTTAVAT MINKÄ YHTÄLÖN ?

DERIVOIDAAN LAGUERREN YHTÄLÖÄ ($\mu = n$) p KERTAA:

$$0 = \frac{d^p}{dz^p}\left(z\frac{d^2\mathcal{L}_n}{dz^2} + (1-z)\frac{d\mathcal{L}_n}{dz} + n\mathcal{L}_n \right) =$$

$$= z\frac{d^2}{dz^2}\mathcal{L}_n^{(p)} + p\frac{d}{dz}\mathcal{L}_n^{(p)} + \frac{d}{dz}\mathcal{L}_n^{(p)} - z\frac{d\mathcal{L}_n^{(p)}}{dz}$$

$$-p\mathcal{L}_n^{(p)} + n\mathcal{L}_n^{(p)} =$$

$$= z\frac{d^2\mathcal{L}_n^{(p)}}{dz^2} + (p+1-z)\frac{d\mathcal{L}_n^{(p)}}{dz} + (n-p)\mathcal{L}_n^{(p)} = 0$$

LAGUERREN LIITTOYHTÄLÖ

ORTONORMITUS

TODISTETAAN ENSIN RODRIGUES-TYYPPINEN KAAVA

$$\mathcal{L}_n(z) = e^z \frac{d^n}{dz^n}\left(z^n e^{-z}\right) \qquad (LR)$$

ON: $$\frac{d^n}{dz^n}\left(z^n e^{-z}\right) = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\frac{n!}{k!(n-k)!}} \underbrace{\frac{d^k e^{-z}}{dz^k}}_{(-1)^k e^{-z}} \underbrace{\frac{d^{n-k}}{dz^{n-k}} z^n}_{\frac{n!}{k!} z^k}$$

$$= e^{-z}\sum_{k=0}^{n} \frac{(-1)^k (n!)^2}{(k!)^2 (n-k)!} z^k = e^{-z}\mathcal{L}_n(z)$$

OSOITETAAN SEURAAVAKSI ETTÄ LAGUERREN POLYNOMIT $\mathcal{L}_n$ OVAT ORTOGONAALISIA POLYNOMEJA VÄLILLÄ $[0,\infty)$ PAINOFUNKTIOLLA $w(x) = e^{-x}$, S.O.

$$\int_0^\infty dx\, e^{-x}\mathcal{L}_m(x)\mathcal{L}_n(x) \propto \delta_{mn}$$

TARKASTELLAAN ($n \geq m$)

$$I_{mn} = \int_0^\infty dx\, e^{-x}\mathcal{L}_m(x)\mathcal{L}_n(x) =$$

$$= \int_0^\infty dx\left(\frac{d^n}{dx^n}\left(x^n e^{-x}\right)\right)\mathcal{L}_m(x) =$$

$$= \underbrace{\Big/_0^\infty\left(\frac{d^{n-1}}{dx^{n-1}}x^n e^{-x}\right)\mathcal{L}_m(x)}_{=0} - \int_0^\infty dx\left(\frac{d^{n-1}}{dx^{n-1}}x^n e^{-x}\right)\frac{d\mathcal{L}_m}{dx}$$

$$= \dots = (-1)^n \int_0^\infty dx\, x^n e^{-x} \underbrace{\frac{d^n \mathcal{L}_m}{dx^n}}_{=0 \text{ JOS } n>m}$$

$\Rightarrow I_{mn} = 0 \qquad m \neq n$

KUN $m = n$ $\quad \frac{d^n \mathcal{L}_n}{dx^n} = \frac{d^n}{dx^n} (-1)^n x^n = (-1)^n n!$

$$\Rightarrow I_{nn} = n! \int_0^\infty dx\, x^n e^{-x} = (n!)^2$$

$$\Rightarrow \int_0^\infty dx\, e^{-x} \mathcal{L}_m(x) \mathcal{L}_n(x) = (n!)^2 \delta_{mn}$$

LAGUERREN FUNKTIOT $\quad F_n(x) = e^{-x/2} \mathcal{L}_n(x)$ OVAT

SIIS ORTOGONAALISIA VÄLILLÄ $[0,\infty)$:

$$\int_0^\infty dx\, F_n(x) F_m(x) = (n!)^2 \delta_{mn}$$

MUODOSTAJAFUNKTIO

FYMM I $\quad f(z)$ ANALYYTTINEN $\Rightarrow \frac{d^n f}{dz^n}(z) = \frac{n!}{2\pi i} \oint_\gamma d\xi \frac{f(\xi)}{(\xi - z)^{n+1}}$

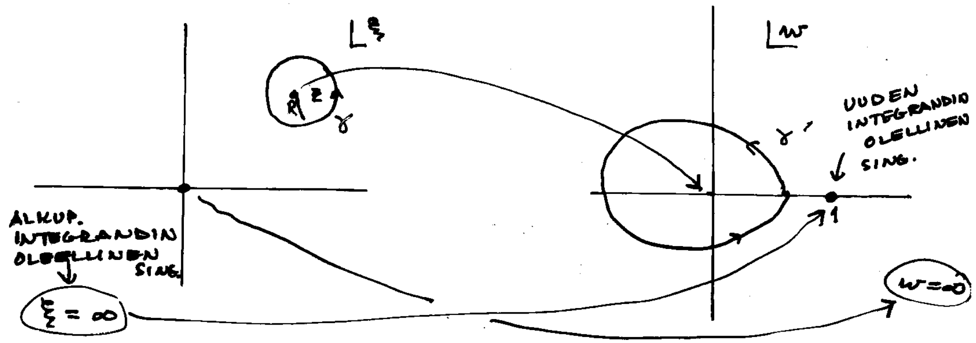
OTETAAN $\quad f(z) = z^n e^{-z}$ $\qquad \cdot z \;\; \gamma \quad L^\xi$

$$(LR) \Rightarrow e^{-z} \mathcal{L}_n(z) = \frac{n!}{2\pi i} \oint_\gamma d\xi \frac{\xi^n e^{-\xi}}{(\xi - z)^{n+1}}$$

$$\frac{1}{n!} \mathcal{L}_n(z) = \frac{1}{2\pi i} \oint_\gamma \frac{d\xi}{\xi} \left(\frac{\xi}{\xi - z} \right)^{n+1} e^{-(\xi - z)}$$

UUSI MUUTTUJA $\quad w = \frac{\xi - z}{\xi} \quad \Leftrightarrow \quad \xi = \frac{z}{1-w}$

$$\frac{d\xi}{\xi} \left(\frac{\xi}{\xi - z} \right)^{n+1} e^{-(\xi - z)} = \frac{dw}{w^{n+1}} \frac{e^{-\frac{zw}{1-w}}}{1-w}$$



ESIM. γ: $\quad \xi - z = R e^{i\theta} \qquad \frac{R}{|z|} < 1 \quad , \; z = |z| e^{+i\alpha}$

$$\Rightarrow \gamma' : \quad w = \frac{1 + \frac{|z|}{R} e^{i\varphi}}{1 + \frac{|z|^2}{R^2} + 2\frac{|z|}{R}\cos\varphi} \qquad \varphi = \theta - \alpha$$

γ' KIERTÄÄ $w = 0$:N KERRAN VASTAPÄIVÄÄN

$$\Rightarrow \frac{1}{n!} \mathcal{L}_n(z) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{dw}{w^{n+1}} \frac{e^{-\frac{zw}{1-w}}}{1-w} = \frac{1}{n!} \frac{\partial^n}{\partial w^n} \left(\frac{e^{-\frac{zw}{1-w}}}{1-w} \right)$$

$$\Rightarrow \boxed{G_0(z,w) \equiv \frac{e^{-\frac{zw}{1-w}}}{1-w} = \sum_{n=0}^\infty \frac{\mathcal{L}_n(z)}{n!} w^n}$$

LIITTOPOLYNOMIEN MUODOSTAJA-FUNKTIO

$$\mathcal{L}_n^{(k)}(z) = \frac{d^k}{dz^k}\mathcal{L}_n(z) \quad ; \; G_0(z,w) = \frac{e^{-\frac{zw}{1-w}}}{1-w}$$

LASKETAAN $$\frac{\partial^k}{\partial z^k}G_0(z,w) = \sum_{n=k}^{\infty}\frac{1}{n}\mathcal{L}_n^{(k)}(z)\,w^n$$

$$\overset{\shortparallel}{\frac{(-1)^k w^k}{(1-w)^{k+1}}\, e^{-\frac{zw}{1-w}}}$$

MÄÄRITELLÄÄN $$G_k(z,w) = \frac{(-1)^k}{(1-w)^{k+1}}\, e^{-\frac{zw}{1-w}}$$

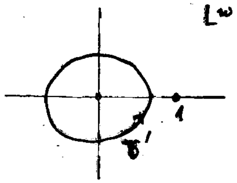
JA SAADAAN $\quad (m = n-k)$

$$\boxed{G_k(z,w) = \sum_{m=0}^{\infty}\frac{1}{(m+k)!}\mathcal{L}_{m+k}^{(k)}(z)\, w^m}$$

JOHDETAAN TÄSTÄ "RODRIGUES-KAAVA" LIITTOPOLYNOMEILLE:

$$\frac{1}{(m+k)!}\mathcal{L}_{m+k}^{(k)}(z) = \frac{1}{m!}\frac{\partial^m}{\partial w^m}G_k(z,w)\Big|_{w=0}$$

$$\lfloor^{w} \; = \frac{1}{2\pi i}\oint_{\gamma'}\frac{dw}{w^{m+1}}\frac{(-1)^k}{(1-w)^{k+1}}\, e^{-\frac{zw}{1-w}}$$



PALAUTUSKAAVOJA:

1) $$\frac{\partial}{\partial w}\left(\frac{e^{-\frac{zw}{1-w}}}{1-w}\right) = \frac{e^{-\frac{zw}{1-w}}}{1-w}\;\frac{1-w-z}{(1-w)^2}\,, \quad \text{ELI}$$

$$\underbrace{(1-w)^2\frac{\partial}{\partial w}G_0(z,w)}_{\sum_{n=0}^{\infty}\frac{\mathcal{L}_{n+1}(z)}{n!}w^n} = (1-w-z)\underbrace{G_0(z,w)}_{\sum_{n=0}^{\infty}\frac{\mathcal{L}_n(z)}{n!}w^n}$$

w^n:N KERROIN MOLEMMIN PUOLIN:

$$\frac{1}{n!}\mathcal{L}_{n+1} - 2\frac{\mathcal{L}_n}{(n-1)!} + \frac{\mathcal{L}_{n-1}}{(n-2)!} = (1-z)\frac{\mathcal{L}_n}{n!} - \frac{\mathcal{L}_{n-1}}{(n-1)!} \quad \Big|\, n!$$

$$\Rightarrow \mathcal{L}_{n+1}(z) - 2n\mathcal{L}_n(z) + n(n-1)\mathcal{L}_{n-1}(z) = (1-z)\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z)$$

ELI

$$\boxed{\mathcal{L}_{n+1}(z) - (2n+1-z)\mathcal{L}_n(z) + n^2\mathcal{L}_{n-1}(z) = 0}$$

2) $$\frac{\partial}{\partial z}\left(\frac{e^{-\frac{zw}{1-w}}}{1-w}\right) = -\frac{w}{1-w}\,\frac{e^{-\frac{zw}{1-w}}}{1-w} \quad \text{ELI}$$

$$\underbrace{(1-w)\frac{\partial}{\partial z}G_0(z,w)}_{\sum_{n=0}^{\infty}\frac{\mathcal{L}_n'(z)}{n!}w^n} = -w\underbrace{G_0(z,w)}_{\sum_{n=0}^{\infty}\frac{\mathcal{L}_n(z)}{n!}w^n}$$

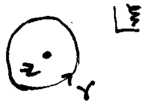
$$\Rightarrow \frac{\mathcal{L}_n'}{n!} - \frac{\mathcal{L}_{n-1}'}{(n-1)!} = -\frac{\mathcal{L}_{n-1}}{(n-1)!}$$

$$\Rightarrow \boxed{\mathcal{L}_n'(z) - n\mathcal{L}_{n-1}'(z) + n\mathcal{L}_{n-1}(z) = 0}$$

MUUTTUJAN VAIHDOS $\quad w \to \xi = \frac{z}{1-w} \quad \Leftrightarrow \quad w = \frac{\xi - z}{\xi}$

(KÄÄNTEINEN ÄSKEISELLE!)

$$\Rightarrow \frac{1}{(m+k)!} \mathcal{L}_{m+k}^{(k)}(z) = \frac{(-1)^k z^{-k} e^{z}}{2\pi i} \oint_\gamma d\xi \, \frac{\xi^{m+k} e^{-\xi}}{(\xi - z)^{m+1}}$$



$$= (-1)^k z^{-k} e^{z} \frac{1}{m!} \frac{d^m}{dz^m}\left(z^{m+k} e^{-z}\right)$$

$$\Rightarrow \boxed{\mathcal{L}_{m+k}^{(k)}(z) = (-1)^k \frac{(m+k)!}{m!} z^{-k} e^{z} \frac{d^m}{dz^m}\left(z^{m+k} e^{-z}\right)}$$

TÄMÄN AVULLA LASKETAAN

$$J_{mn}^{(k)} = \int_0^\infty dx \, e^{-x} x^k \mathcal{L}_{m+k}^{(k)}(x) \mathcal{L}_{n+k}^{(k)}(x)$$

OLKOON $m \geq n$

$$J_{mn}^{(k)} = (-1)^k \frac{(m+k)!}{m!} \int_0^\infty dx \left(\frac{d^m}{dx^m}\left(x^{m+k} e^{-x}\right)\right) \mathcal{L}_{n+k}^{(k)}(x)$$

$$= (-1)^k \frac{(m+k)!}{m!} \Big/_0^\infty \frac{d^{m-1}}{dx^{m-1}}\left(x^{m+k} e^{-x}\right) \mathcal{L}_{n+k}^{(k)} \leftarrow = 0$$

$$+ (-1)^{k+1} \frac{(m+k)!}{m!} \int_0^\infty dx \, \frac{d^{m-1}}{dx^{m-1}}\left(x^{m+k} e^{-x}\right) \frac{d\mathcal{L}_{n+k}^{(k)}}{dx}$$

$$= \cdots = (-1)^{k+m} \frac{(m+k)!}{m!} \int_0^\infty dx \, x^{m+k} e^{-x} \frac{d^m \mathcal{L}_{n+k}^{(k)}}{dx^m}$$

JOS $m > n$, $\quad \frac{d^m \mathcal{L}_{n+k}^{(k)}}{dx^m} = 0$

$$m = n \quad \frac{d^n \mathcal{L}_{n+k}^{(k)}}{dx^n} = \frac{d^{n+k}}{dx^{n+k}} (-1)^{n+k} x^{n+k} = (-1)^{n+k} (n+k)!$$

$$\Rightarrow J_{nn}^{(k)} = (-1)^{2(n+k)} \frac{((n+k)!)^2}{n!} \int_0^\infty dx \, x^{n+k} e^{-x}$$

$$= \frac{[(n+k)!]^3}{n!}$$

SIIS

$$\boxed{\int_0^\infty dx \, e^{-x} x^k \mathcal{L}_{m+k}^{(k)}(x) \mathcal{L}_{n+k}^{(k)}(x) = \frac{[(n+k)!]^3}{n!} \delta_{mn}}$$

LAGUERREN LIITTOPOLYNOMIT $\mathcal{L}^{(k)}$ OVAT ORTOGONAALISIA VÄLILLÄ $[0,\infty)$ PAINOFUNKTIOLLA

$$w(x) = x^k e^{-x}$$

VETYATOMIN RADIAALINEN YHTÄLÖ

SCHRÖDINGER:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi + V\psi = E\psi$$

μ HIUKKASEN MASSA (TÄSSÄ ELEKTRONIN)

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

YRITE $\quad \psi(r,\theta,\varphi) = R(r) Y_{lm}(\theta,\varphi)$

$\Rightarrow$ RADIAALINEN YHTÄLÖ

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)R + \frac{\hbar^2}{2\mu r^2} l(l+1)R - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} R = E R$$

KIRJ. $R(r) = \frac{u(r)}{r}$

$$\Rightarrow \frac{d^2u}{dr^2} + \left[-\frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{2\mu E}{\hbar^2} \right] u = 0$$

MÄÄR. $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137{,}04}$ JA DIMENSIOTTOMAT SUUREET

HIENORAKENNEVAKIO

$\xi = \frac{\mu c}{\hbar} r \equiv \frac{r}{\lambda_c}$ $\lambda_c = \frac{\hbar}{\mu c} \simeq 2{,}426 \times 10^{-12}\,\mathrm{m}$ ELEKTRONIN COMPTON-AALLONPITUUS

$\epsilon = \frac{2E}{\mu c^2}$

$$\Rightarrow \frac{d^2u}{d\xi^2} + \left(\varepsilon + \frac{2\alpha}{\xi} - \frac{l(l+1)}{\xi^2} \right) u = 0$$

VIELÄ: $\rho = 2\sqrt{-\varepsilon}\,\xi$ ($\varepsilon < 0$ SIDOTUILLE TILOILLE)

JA $u(\rho) = C \rho^{l+1} e^{-\rho/2} f(\rho)$

$\Rightarrow f$ TOTEUTTAA

$$\rho \frac{d^2f}{d\rho^2} + (2l+2-\rho)\frac{df}{d\rho} + \left(\frac{\alpha}{\sqrt{-\varepsilon}} - l - 1 \right) f = 0$$

LAGUERREN LIITTOYHTÄLÖ $p = 2l+1$, $n = \frac{\alpha}{\sqrt{-\varepsilon}} + l$

PÄÄKVANTTILUKU $N = n - l$

$$\varepsilon_N = -\frac{\alpha^2}{N^2} \Rightarrow E_N = -\frac{1}{2}\mu c^2 \frac{\alpha^2}{N^2}$$

HERMITEN POLYNOMIT

(CHARLES HERMITE 1822 - 1901)

MUODOSTAJAFUNKTIO:

$$g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

HERMITEN POLYNOMIT

$H_0(x) = g(x,0) = 1$

$H_1(x)\,t = 2xt \Rightarrow H_1(x) = 2x$

JNE

PALAUTUSKAAVOJA:

$$\frac{\partial}{\partial x} g(x,t) = 2t\,g(x,t) = 2\sum_{n=1}^{\infty} n H_{n-1} \frac{t^n}{n!}$$

$$\overset{\shortparallel}{\sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}}$$

$$\Rightarrow \boxed{H_n'(x) = 2n H_{n-1}(x)} \quad \text{(P1)}$$

$$g(x,t) = e^{-t^2+2xt} = e^{-(t-x)^2} \cdot e^{x^2}$$

$$\partial_t g(x,t) = 2(x-t)\, g(x,t) = 2\sum_{n=0}^{\infty}\left(x H_n(x) - n H_{n-1}(x)\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} H_{n+1}(x)\, \frac{t^n}{n!}$$

$$\Rightarrow \boxed{H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)} \qquad (P2)$$

$$\Rightarrow H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x \qquad \text{JNE.}$$

TOINEN ESITYS:

$$(G) \Rightarrow H_n(x) = \frac{\partial^n}{\partial t^n} g(x,t)\Big|_{t=0} = e^{x^2}\frac{\partial^n}{\partial t^n} e^{-(t-x)^2}\Big|_{t=0}$$

$$= e^{x^2}\frac{\partial^n}{\partial(-x)^n} e^{-(t-x)^2}\Big|_{t=0} = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

$$\Rightarrow \boxed{H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n} e^{-x^2}} \qquad \text{("RODRIGUES")}$$

DIFFERENTIAALIYHTÄLÖ:

$$H_n'' \overset{(P1)}{=} 2n H_{n-1}' \overset{(P2)}{=} \frac{d}{dx}\left(2x H_n - H_{n+1}\right) = 2H_n + 2x H_n' - H_{n+1}' =$$

$$\overset{(P1)}{=} 2H_n + 2x H_n' - 2(n+1) H_n = 2x H_n' - 2n H_n$$

SIIS:

$$\boxed{\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2n H_n = 0}$$

INTEGRAALIKAAVA

OLKOON $\Pi_m(x) = A_m x^m + A_{m-1}x^{m-1} + \dots + A_0$ POLYNOMI ASTETTA m

HALUAMME LASKEA

$$I_{mn} = \int_{-\infty}^{\infty} dx\; e^{-x^2}\,\Pi_m(x)\, H_n(x) \qquad \text{KUN } m \le n$$

KÄYTETÄÄN "RODRIGUES-KAAVAA" $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$

$$\Rightarrow I_{mn} = (-1)^n \int_{-\infty}^{\infty} dx\; \Pi_m(x)\frac{d^n}{dx^n}e^{-x^2} = (-1)^n \Big/_{-\infty}^{\infty} \underbrace{\Pi_m(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}}_{\text{Polynomi} \times e^{-x^2}}$$

$\Rightarrow$ SIJOITUSTERMI $= 0$

$$+ (-1)^{n+1}\int_{-\infty}^{\infty} dx\; \frac{d\Pi_m}{dx}\frac{d^{n-1}e^{-x^2}}{dx^{n-1}} =$$

$$= \dots = (-1)^{2n}\int_{-\infty}^{\infty} dx\; \frac{d^n \Pi_m}{dx^n}\, e^{-x^2}$$

JOS $m < n$ $\quad \frac{d^n \Pi_m}{dx^n} = 0$, JA $I_{mn} = 0$

JOS $m = n$ $\quad \frac{d^n \Pi_n}{dx^n} = A_n n!$, $\quad \int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$

$$\rightarrow I_{nn} = A_n n! \sqrt{\pi}$$

ERIKOISESTI, JOS $\Pi_m(x) = H_m(x)$:

"RODRIGUES" $H_n(x) = (-1)^n(-2x)^n$ + ALEMMAN KERTALUVUN TERMEJÄ

$$\Rightarrow A_n = 2^n$$

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x) H_n(x) = 2^n n! \sqrt{\pi}\, \delta_{mn}}$$

HERMITEN FUNKTIOT $\varphi_n(x) = e^{-x^2/2} H_n(x)$

TOTEUTTAVAT :

* DIFFERENTIAALIYHTÄLÖN

$$\frac{d^2\varphi_n}{dx^2} + (2n+1 - x^2)\varphi_n = 0$$

$$\Big(H_n = e^{x^2/2}\varphi_n \Rightarrow H_n' = e^{x^2/2}(\varphi_n' + x\varphi_n)$$

$$H_n'' = e^{x^2/2}(\varphi_n'' + 2x\varphi_n' + x^2\varphi_n + \varphi_n)$$

$$\Rightarrow H_n'' - 2xH_n' + 2nH_n = e^{x^2/2}\big(\varphi_n'' + 2x\varphi_n' + x^2\varphi_n + \varphi_n$$

$$- 2x\varphi_n' - 2x^2\varphi_n + 2n\varphi_n\big) = e^{x^2/2}\big(\varphi_n'' + (2n+1-x^2)\varphi_n\big) = 0 \Big)$$

* ORTOGONAALISUUSEHDON

$$\int_{-\infty}^{\infty} dx\, \varphi_m(x)\varphi_n(x) = \sqrt{\pi}\, 2^n n!\, \delta_{mn}$$

HERMITEN FUNKTIOT OVAT KVANTTIMEKAANISEN HARMONISEN OSKILLAATTORIN ENERGIAN OMINAISFUNKTIOT :

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \qquad \left(\hat{p} = -i\hbar\frac{d}{dx}\right)$$

AJASTA RIIPPUMATON SCHRÖDINGERIN YHTÄLÖ

$$H\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2\psi = E\psi$$

EHTO : $\int dx\, |\psi(x)|^2 < \infty$

SIIRRITÄÄN DIMENSIOTTOMAAN MUUTTUJAAN

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\Big([\hbar] = \frac{ML^2}{T}, \quad [\omega] = \frac{1}{T}$$

$$\Rightarrow \left[\frac{m\omega}{\hbar}\right] = \frac{M}{T}\frac{T}{ML^2} = \frac{1}{L^2} \Rightarrow \left[\sqrt{\frac{m\omega}{\hbar}}\,x\right] = 1 \quad \text{OK} \Big)$$

$$\frac{d^2}{dx^2} = \frac{m\omega}{\hbar}\frac{d^2}{d\xi^2}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = \frac{\hbar\omega}{2}\left(-\frac{d^2\psi}{d\xi^2} + \xi^2\psi\right) = E\psi$$

MERK. $\lambda = \frac{2E}{\hbar\omega}$ VRT $\varphi_n'' + (2n+1-x^2)\varphi_n = 0$

$$\Rightarrow \frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0$$

$$\Rightarrow \psi_n(\xi) = N_n\,\varphi_n(\xi) \qquad \lambda_n = 2n+1$$

NORMITUSVAKIO

$$\Rightarrow E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

'TIKAPUUOPERAATTORIT'

$$\left(\xi - \frac{d}{d\xi}\right)\varphi_n(\xi) = e^{-\xi^2/2}\left(\xi H_n(\xi) + \xi H_n(\xi) - H_n'(\xi)\right)$$

$$\overset{(P1)}{=} e^{-\xi^2/2}\left(2\xi H_n(\xi) - 2n H_{n-1}(\xi)\right) \overset{(P2)}{=} e^{-\xi^2/2} H_{n+1}(\xi)$$

$$\Rightarrow \varphi_{n+1}(\xi) = \left(\xi - \frac{d}{d\xi}\right)\varphi_n(\xi)$$

$$\left(\xi + \frac{d}{d\xi}\right) \varphi_n(\xi) = e^{-\xi^2/2} \left(\cancel{\xi H_n} - \cancel{\xi H_n} + H_n' \right) \overset{(A)}{=}$$

$$= 2n\, e^{-\xi^2/2} H_{n-1}(\xi) = 2n\, \varphi_{n-1}(\xi)$$

$$\Rightarrow \varphi_{n-1}(\xi) = \frac{1}{2n} \left(\xi + \frac{d}{d\xi}\right) \varphi_n(\xi)$$

$\varphi_{-1} \equiv 0$ PERUSTILA φ_0 SAADAAN RATKAISEMALLA

$$\left(\xi + \frac{d}{d\xi}\right) \varphi_0(\xi) = 0$$

$$\Rightarrow \varphi_0 = N\, e^{-\xi^2/2}$$